International Journal of Theoretical Physics, Vol. 8, No. 4 (1973), pp. 243–246

Electromagnetic Test Fields around a Schwarzschild Singularity

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Received: 14 *December* 1972

Abstract

If the two invariants of an arbitrary non-static electromagnetic vacuum field are finite at the Schwarzschild radius $r = M$, the field behaves at $r = M_+$ either as a purely ingoing or as a purely outgoing wave.

1. *Statement of the Problem*

It is usually said that a light ray (or photon) can pass from the exterior world into the Schwarzschild singularity, but can never escape from a black hole. The aim of this paper is to ask the full set of Maxwell equations, and not only geometrical optics, what they say about this problem. To get a clear answer we have to impose a regularity condition: In agreement with the fact that the invariants of the gravitational field, e.g. $(-g)^{1/2}$ and scalar curvature R, are finite at $r = M$, we admit Maxwell fields with finite invariants only. The technique used in this paper is that of Debye potentials.

2. *Debye Potentials*

The Schwarzschild metric

$$d\bar{s}^2 = \frac{r}{r-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) - \frac{r-1}{r} dt^2 \tag{2.1}$$

is conformally equivalent to

$$ds^2 = \frac{r^3}{r-1}(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) + dv^2 - dt^2, \qquad v = r + \ln(r-1) \tag{2.2}$$

To get these and the subsequent formulas in the usual units one has to replace r by r/M, v by v/M and, later on, ω by ω/M.

By means of Debye potentials (Stephani, 1973) it is possible to get the four-potential A_n of an arbitrary non-static electromagnetic field by solving the Debye equation

$$D(\pi) = \frac{r-1}{r^3}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial\pi}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2\pi}{\partial\varphi^2}\right] + \frac{\partial^2\pi}{\partial v^2} - \frac{\partial^2\pi}{\partial t^2} = 0 \quad (2.3)$$

for π and ϕ and inserting them into

$$\begin{aligned} A_a &= \pi_{,n}(u^n v_a - v^n u_a) + \varepsilon_a^{bpq}\phi_{,b} v_p u_q \\ v^a &= (0,0,1,0), \qquad u^a = (0,0,0,1) \end{aligned} \quad (2.4)$$

Formula (2.4) holds in the metric (2.2). Metric and electromagnetic fields of the two spaces (2.1) and (2.2) are related by

$$\begin{aligned} d\bar{s}^2 &= \frac{r-1}{r}\, ds^2, \qquad F_{\bar{a}\bar{b}} = F_{ab} = A_{b,a} - A_{a,b} \\ A_{\bar{a}} &= A_a, \qquad F^{\bar{a}\bar{b}} = F^{ab}\frac{r^2}{(r-1)^2} \end{aligned} \quad (2.5)$$

3. *Behaviour of Debye Potentials at $r = 1$*

The general solution of the Debye equation may be written as

$$\pi = \int d\omega \sum_{n,m} A_{nm}(\omega)\, Y_n^m(\vartheta,\varphi)\, R_n(r)\, e^{-i\omega t} \quad (3.1)$$

where $A_{nm}(\omega)$ are arbitrary functions of ω, $Y_n^m(\vartheta,\varphi)$ are the usual surface harmonics and R_n is a solution of the differential equation

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r(r-1)}\frac{dR_n}{dr} + \left[\frac{\omega^2 r^2}{(r-1)^2} - \frac{n(n+1)}{r(r-1)}\right] R_n = 0 \quad (3.2)$$

investigated by Whittaker (1927).

The point $r = 1$ is a regular singular point (Ince, 1956) of equation (3.2). The ansatz

$$R_n = (r-1)^\rho . \sum_{\nu=0}^{\infty} c_\nu (r-1)^\nu \quad (3.3)$$

gives the condition

$$\rho^2 = -\omega^2 \quad (3.4)$$

which is independent of n. Because of $(r-1)^{\pm i\omega} = e^{\pm i\omega \ln(r-1)}$, it follows that the general solution of (3.2) has the structure

$$R_n(r) = e^{-i\omega v} R_n^-(r) + e^{i\omega v} R_n^+(r) \quad (3.5)$$

R_n^- and R_n^+ being regular functions at $r = 1$.

The main result of this analysis is that the Debye potentials of a non-static field—$A_{nm}(0) = 0$—

$$\begin{aligned} \pi(r, \vartheta, \varphi, t) &= \int [\pi^+(r, \vartheta, \varphi, \omega)\,\mathrm{e}^{i\omega v} + \pi^-(r, \vartheta, \varphi, \omega)\,\mathrm{e}^{-i\omega v}]\,\mathrm{e}^{-i\omega t}\,d\omega \\ \phi(r, \vartheta, \varphi, t) &= \int [\phi^+(r, \vartheta, \varphi, \omega)\,\mathrm{e}^{i\omega v} + \phi^-(r, \vartheta, \varphi, \omega)\,\mathrm{e}^{-i\omega v}]\,\mathrm{e}^{-i\omega t}\,d\omega \end{aligned} \tag{3.6}$$

are finite at $r = 1$, because $\pi^\pm$ and $\phi^\pm$ are regular functions of $r - 1$. Due to the rapidly oscillating factors $\mathrm{e}^{\pm i\omega v}$ the derivatives of π and ϕ with respect to r become infinite while approaching $r - 1$.

4. *Condition of Finite Field Invariants*

Using the notations

$$\begin{aligned} a_1 &= a(\pi) = \frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial\pi}{\partial\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial^2\pi}{\partial\varphi^2}, & a_2 &= a(-\phi) \\ b_1 &= b(\pi, \phi) = -\sin\vartheta\frac{\partial^2\pi}{\partial v\,\partial\vartheta} - \frac{\partial^2\phi}{\partial\varphi\,\partial t}, & b_2 &= b(-\phi, \pi) \\ c_1 &= c(\pi, \phi) = -\sin\vartheta\frac{\partial^2\pi}{\partial t\,\partial\vartheta} - \frac{\partial^2\phi}{\partial\varphi\,\partial v}, & c_2 &= c(-\phi, \pi) \end{aligned} \tag{4.1}$$

the invariants of the electromagnetic field are

$$\begin{aligned} F_{\overline{mn}}F^{\overline{mn}} &= \frac{2}{(r-1)r\sin^2\vartheta}\left[c_1{}^2 - c_2{}^2 - b_1{}^2 + b_2{}^2 - \frac{r-1}{r^3}(a_1{}^2 - a_2{}^2)\right] \\ \tilde{F}_{\overline{mn}}F^{\overline{mn}} &= -\frac{4}{(r-1)r\sin^2\vartheta}\left[c_1c_2 - b_1b_2 - a_1a_2\frac{r-1}{r^3}\right] \end{aligned} \tag{4.2}$$

In (4.1), (4.2) and (4.3) π and ϕ and therefore $a_1, \ldots, c_2$ should be taken in a real (not complex) representation.

The functions a_1 and a_2 are finite at $r = 1$. So the admitted fields have to fulfil the condition

$$(c_1 + ic_2)^2 = (b_1 + ib_2)^2 \tag{4.3}$$

or

$$c_1 = \varepsilon b_1, \qquad c_2 = \varepsilon b_2, \qquad \varepsilon = \pm 1 \tag{4.4}$$

at $r = 1$. Due to the fact that (4.4) has to be valid for each frequency ω and that the coefficients of $\mathrm{e}^{i\omega v}$ and $\mathrm{e}^{-i\omega v}$ have to balance separately, (4.4) is equivalent to the four equations

$$\begin{aligned} (\varepsilon + 1)\left(\sin\vartheta\frac{\partial\pi^+}{\partial\vartheta} - \frac{\partial\phi^+}{\partial\varphi}\right) &= 0 \\ (\varepsilon + 1)\left(\sin\vartheta\frac{\partial\phi^+}{\partial\vartheta} + \frac{\partial\pi^+}{\partial\varphi}\right) &= 0 \end{aligned}$$

$$(\varepsilon - 1)\left(\sin\vartheta\frac{\partial\phi^-}{\partial\vartheta} - \frac{\partial\pi^-}{\partial\varphi}\right) = 0$$

$$(\varepsilon - 1)\left(\sin\vartheta\frac{\partial\pi^-}{\partial\vartheta} + \frac{\partial\phi^-}{\partial\varphi}\right) = 0 \tag{4.5}$$

Debye potentials π and ϕ, which fulfil the Debye equation (2.3) and are independent of ϑ and φ, give no contribution to the fields F_{ab}. So we get from (4.5) the two possible solutions

$$\pi^+ = 0, \qquad \phi^+ = 0, \qquad \varepsilon = 1 \tag{4.6}$$

and

$$\pi^- = 0, \qquad \phi^- = 0, \qquad \varepsilon = -1 \tag{4.7}$$

5. *Discussion*

The result of our considerations, given in (4.6), (4.7) and (3.6) is: If we approach the Schwarzschild radius from outside the black hole, the general non-static Maxwell field behaves as a purely ingoing wave ($\varepsilon = 1$) or as a purely outgoing wave ($\varepsilon = -1$). A mixture of both types is not admitted. To avoid confusion it should be stated explicitly that in both cases the field at infinity is a superposition of in- and outgoing waves.

The existence of the outgoing wave ($\varepsilon = -1$), i.e. of a black hole emitting radiation, cannot be excluded by a regularity condition. If we are not willing to accept it we have to impose additional conditions concerning, e.g., initial values, cosmological models or sources of the field inside $r = 1$.

The regular static field has been given by Israel (1968). It is a superposition of a spherically symmetric vacuum field and the field of arbitrary sources situated outside $r = 1$.

Acknowledgement

We have to thank all members of our Jena group for valuable discussions.

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